Determination of Free-Body Responses from Constrained Tests

ALEX BERMAN* AND NICHOLAS GIANSANTE†
Kaman Aerospace Corporation, Bloomfield, Conn.

The method is based on the principle that a constrained structure can be considered to be a free body acted upon by multiple forces which include the forces of constraint. By measuring these forces and by exciting the structure so as to develop linearly independent sets of forces, one can compute the response of the free body to one force at a time. Techniques for producing these independent forces are discussed. Computer simulations of tests (including experimental error) are presented. The procedure appears to be a feasible approach to obtaining the in-flight characteristics of aerospace vehicles.

Nomenclature

f = applied force
$\mathbf{f}_k$ = complex force vector
$\mathbf{F}$ = matrix of forces
$\mathbf{F}_S$ = matrix of forces due to applied forces at the constraints
$\mathbf{F}_{SA}$ = matrix of constraint forces due to excitation at other points
M = number of points at which force is applied
N = number of points at which responses are measured
r_i = reaction force at constraint i
$\mathbf{y}_k$ = vector of measured responses
$\mathbf{y}$ = matrix of responses
$\mathbf{Y}$ = mobility matrix relating forces and responses
ω = frequency of applied sinusoidal forces
[ ] = matrix
{ } = column vector

Introduction

DYNAMIC testing, both full scale and model, is an essential step in predicting the response of aerospace vehicles to the conditions to which they will be subjected in flight. This testing is required for modal analysis, stability and control studies, and loads analyses and is applied to design verification and modification studies. The actual in-flight boundary conditions, however, cannot be exactly duplicated on the ground.

In order to simulate the free-body boundary conditions of a vehicle in flight, the usual procedure has been to support the vehicle on a system which is relatively soft so that the "rigid body" frequencies (which should be zero) are low compared to the frequencies of the deformation modes of the structure. A commonly used technique[1-3] for launch vehicles consists of supporting the vehicle vertically on cables attached to its base. Although tests conducted in such a manner seem to have given acceptable results, there are several disadvantages to this scheme. It is necessary to construct a tall structure capable of supporting the total weight of the vehicle. There is some uncertainty in the effects of the cable dynamics and nonlinearities on the vehicle response.[2] Various cable configurations have been known to give variation in test results.[1] Certain new problems arise for vehicles which are not axisymmetric. When the center of gravity varies laterally under various fuel loads, the stabilization of such a vehicle on soft supports can become a major consideration. Vehicles which require testing in more than one attitude compound these difficulties.

A procedure which could eliminate the effects of supports would be of significant benefit. It would not be necessary to use soft suspensions with the assumption that the interactions with the supporting structure are not significant. It would be possible to support the system being tested on a relatively stiff base, thus simplifying the problems of static stability and attitude variation. The design of supporting towers would be greatly simplified and the over-all cost of testing would be reduced. It is essential, of course, that such a procedure be reliable, accurate, not overly sensitive to measurement errors, and applicable to real test conditions.

There are several analytical methods which convert constrained responses into free-body responses. Typical methods are presented in Refs. 4–6. These methods, however, are suitable only for analytical procedures where the response on infinitely rigid supports is known (or can be calculated) and where the mass matrix of the structure is available. Since such data is unmeasurable in a test, these methods are not usable.

The method described in the paper uses the measured forces of constraint to convert the measured structural responses to free-body responses. The structure under test is considered to be supported on real supports, but their specific characteristics are not required since only their measured reactions are used. The procedure uses only data which are actually measured, and no quantitative assumptions are used.

The basic principle on which this method is based was first suggested in Ref. 7. In that work a technique was developed which was appropriate to nondestructive flutter testing and for determining approximate natural frequencies of a structure. The approach taken in this paper is aimed at the determination of the response of a free structure at any frequency. This technique was first discussed in Ref. 8 where a simulated application was made to a beam representation of a helicopter fuselage. In this paper, an application is made to a space shuttle model with a consideration of realistic error effects. The sensitivity to error is considered to be one of the most critical considerations in the determination of the practicality of the technique.

Presented as Paper 73-191 at the AIAA 11th Aerospace Sciences Meeting, Washington, D.C., January 10–12, 1973; submitted January 11, 1973; revision received May 24, 1973. The authors are particularly indebted to the editors for calling their attention to Ref. 7. The data presented were obtained under NASA Contract NAS 1-11535 for the Langley Research Center, Hampton, Va.

Index categories: Aircraft Testing (Including Component Wind Tunnel Testing); Spacecraft Ground Testing and Simulation (Including Components); Structural Dynamic Analysis.

* Principal Research Engineer. Member AIAA.
† Research Specialist.

Description of the Theory

Basic Concept

Consider a constrained structure which is being shaken by a known force and assume that the reaction forces at the supports are known. The structure responds precisely as if it were a free body being simultaneously subjected to the actual applied forces and to the forces of constraint. Thus, a shake test in which the constraining forces are measured gives direct information about the free body response of the structure when acted upon by several forces. As will be seen below, it is possible to convert information of this type into the response of the free body to one force at a time. This is what is needed to determine resonance data and to predict the effects of arbitrary loads.

Analysis

For sinusoidal forces at a frequency ω applied at M points on a structure, there is defined a vector, $\bar{\mathbf{f}}_k$, which represents the complex amplitude of applied force at each of the points. Similarly, $\bar{\mathbf{y}}_k$ is a vector representing the complex amplitudes of the deflection at each of N points resulting from the force, $\bar{\mathbf{f}}_k$. There is no necessity for the force points (represented in $\bar{\mathbf{f}}_k$) to coincide with the response points (represented in $\bar{\mathbf{y}}_k$). The $N \times M$ matrix $\mathbf{Y}$ is the complex displacement mobility and represents the relationship between the forces and responses. The relationship between these quantities may be written

$$\bar{\mathbf{y}}_k = \mathbf{Y}\bar{\mathbf{f}}_k \tag{1}$$

where

$$\bar{\mathbf{y}}_k = \{y_1 \ldots y_N\}, \qquad \bar{\mathbf{f}}_k = \{f_1 \ldots f_M\}$$

The displacement is used only for illustration; exactly the same relationships hold for velocity and acceleration. The displacements can also, with no change in the analysis, represent displacements or rotations in two or three directions at one geometrical point by allowing one element in each vector for each of these generalized displacements. Similar considerations apply to the forces (or moments). Note that there is no necessity for $\mathbf{Y}$ to be square; it will contain one row for each displacement measured and one column for each point at which a force is applied and, as will be seen below, one column for each constraint.

Consider, now, a matrix $\mathbf{F}$ containing several applied load vectors and a matrix $\mathbf{y}$ containing the corresponding deflections as follows:

$$\mathbf{F} = [\bar{\mathbf{f}}_1\bar{\mathbf{f}}_2\ldots] \qquad \mathbf{y} = [\bar{\mathbf{y}}_1\bar{\mathbf{y}}_2\ldots]$$

and then

$$\mathbf{y} = \mathbf{YF} \tag{2}$$

If $\mathbf{F}$ is a nonsingular matrix, then the desired result, the response of particular coordinates to single forces, may be written

$$\mathbf{Y} = \mathbf{yF}^{-1} \tag{3}$$

where both $\mathbf{y}$ and $\mathbf{F}$ are measured. When the "actual" applied loads only are included in $\mathbf{F}$, then $\mathbf{Y}$ is the mobility of the structure as tested, i.e., on the actual supports. *If* $\mathbf{F}$ *includes any of the forces of constraint, then* $\mathbf{Y}$ *is the mobility of the structure with those constraints removed. If* $\mathbf{F}$ *includes all the forces of constraint, then* $\mathbf{Y}$ *is the mobility of the free body.*

As stated previously, $\mathbf{F}$ must be nonsingular and thus have an inverse. If there are M forces to be considered (including the forces at the constraints) then M sets of forces, $\bar{\mathbf{f}}_k$, must be applied and all of these vectors must be independent. There are at least two ways that this may be done: 1) by applying an external force at each constraint, or 2) by varying the constraints.

Forces at Constraints

If an exciting force is applied at the kth constraint, the force vector will be of the form

$$\bar{\mathbf{f}}_k = \{r_1 r_2 \ldots r_k + f \ldots\} \tag{4}$$

where the r's are the measured forces of constraint and f is the applied force. The force vectors obtained by applying forces at each of the constraints will ordinarily be independent of each other and the force matrix will therefore be nonsingular. The possibility of an ill-conditioned matrix is discussed in the section on illustrative computations. These vectors are then formed into a matrix of forces at the supports, including the "actual" applied forces. This matrix will be called $\mathbf{F}_S$.

At the same time that these forces are measured, the displacements are measured at the points of interest on the structure and one column of y is formed for each column of $\mathbf{F}_S$. Then, as aforementioned

$$\mathbf{Y} = \mathbf{yF}_S^{-1} \tag{5}$$

where $\mathbf{Y}$ represents the deflection of each point of interest due to each of the applied loads (at the supports). This is the free-body mobility matrix. This procedure must be carried out over the frequency range of interest.

If it is desired to find the response due to forces at points other than the supports, then the structure must be shaken at these points in addition and the forces at the constraints must be recorded. If $\mathbf{F}_{SA}$ is a matrix representing the forces of constraint for each excitation not at a support, then the $\mathbf{F}$ matrix becomes

$$\mathbf{F} = \begin{bmatrix} \mathbf{I} & \vdots & 0 \\ \cdots & \cdots & \cdots \\ \mathbf{F}_{SA} & \vdots & \mathbf{F}_S \end{bmatrix} \tag{6}$$

where unit forces are applied. The inverse of this matrix involves little more than inverting $\mathbf{F}_S$ and can be obtained by

$$\mathbf{F}^{-1} = \begin{bmatrix} \mathbf{I} & \vdots & 0 \\ \cdots & \cdots & \cdots \\ -\mathbf{F}_S^{-1}\mathbf{F}_{SA} & \vdots & \mathbf{F}_S^{-1} \end{bmatrix} \tag{7}$$

or by direct numerical inversion of $\mathbf{F}$

Varied Constraints

Any means of varying the constraint forces such that the $\mathbf{F}$ matrix is nonsingular will work. Applying a force at each of the constraints was just discussed. Another method is to vary the constraints themselves such that the force vectors are independent.

If the structure in supported redundantly, then a procedure which would work is to shake at any one constraint and remove one constraint at a time resulting in an F matrix of the following form:

$$F = \begin{bmatrix} r_1+f & r_1+f & r_1+f & 0+f\ldots \\ r_2 & r_2 & 0 & r_2\ldots \\ r_3 & 0 & r_3 & r_3\ldots \\ \vdots & & \vdots & \end{bmatrix} \tag{8}$$

where the first column represents the measured loads when all the supports are used, the second represents the loads with constraint number 3 removed, etc.

The same effect can be achieved by varying some parameters, e.g., the stiffness, of each constraint one at a time. This would eliminate the need for redundant supports and reduce the amount of data required. The other considerations are similar to the previous method.

Practical Considerations

The method has attributes which make it an especially attractive candidate for practical application including the use of only measured data and the lack of quantitative assumptions. There are, however, as in all procedures, certain considerations involved in planning an efficient and accurate application of the method.

Number of Constraints

At each frequency, it is necessary to conduct one test for each constraint, thus it is desirable to keep this number to a minimum. Although it is possible to constrain all rigid body motions with six constraining forces, there is no necessity for such complete constraint. During the design of a test, consideration should be given to test configurations which allow freedom of motion, e.g., in the horizontal plane and around the vertical axis. In this case, it would be necessary to shake vertically at each support and measure each of the vertical forces of constraint. In addition, any other shaker position or orientation could be used while the vertical forces were measured.

For the design of a specific test, it is necessary to evaluate the cost of eliminating constraints compared to the reduced testing required.

Support Characteristics

Theoretically the characteristics of the supports are immaterial. These characteristics, however, do affect the magnitudes of the

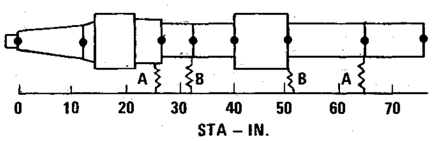


Fig. 1 Orbiter model with two sets of constraints used in the simulated tests.

forces and displacements which will be measured. The various transducers (accelerometers, load cells, etc.) perform more or less accurately depending on the magnitude and frequency of the quantity being measured. Thus, for the most reliable results, the supports should be designed and the transducers selected so as to operate the transducers in their most accurate region. It is not expected that this will be an extremely stringent requirement, but care must be exercised as in planning any test, to insure that the results be meaningful.

Errors

One of the most important considerations is the sensitivity of the method to experimental error. The process uses a measured force matrix containing errors, inverts this matrix and multiplies by another measured matrix also containing errors. The behavior of these errors will determine whether the method is economically feasible. The expected accuracy of the final results compared to the accuracy of an alternate method is an important consideration. It is possible that a final tradeoff must be considered between the cost of improved instrumentation and the savings accrued from avoiding the necessity for soft supports.

Test Simulation

A computer simulation of alternative test configurations can be an extremely useful tool in the preliminary design of any test. Such a simulation applied to this method using an approximate analytical model of the vehicle can be used to determine the sensitivity to error and the expected accuracy of the results of the various arrangements considered. It should include realistic experimental errors, approximate constraint characteristics and vary the frequency over the range of interest.

Simulation Software

Computer Program Organization

In order to evaluate the concepts just presented, a computer program was written to yield simulated test data which would be as realistic as possible.

The computer program consists of three parts. Part one forms the complete mobility matrix of the system at specified frequencies. An analytical finite element model of the structure is used. Also, the constraint characteristics, including location, stiffness, and structural or viscous damping are included. (It should be noted that in a real test, these parameters will not be known.) Part two converts the mobility matrices generated in part one to simulated measured constrained acceleration data including the effects of measurement and system error. In addition, the measured force matrix (also including error) is calculated. Part three of the program calculates the mobility of the free body by multiplying the simulated measured acceleration response matrix, by the inverted force matrix. All computations are performed using complex arithmetic. Part three has been written so as to be capable of analyzing actual test data.

Simulated Errors

The program handles several types of errors simultaneously applied to both the simulated measured accelerations and the simulated measured forces. These errors include: 1) a uniformly distributed random percentage error on amplitude between specified limits; 2) a uniformly distributed random phase angle error between specified limits; 3) a constant specified percentage bias error; and 4) a uniformly distributed random amplitude error between limits (simulating system noise).

Illustrative Computations

The Model

The structure analyzed for the illustrative analyses presented here is a 16-degree-of-freedom representation of a $\frac{1}{15}$ scale model of the orbiter fuselage of a space shuttle configuration.[9] Each of the eight coordinates was allowed a rotational and a transverse degree of freedom. Two different constraints were used for these illustrations. The model is illustrated in Fig. 1, where the locations of the two sets of constraints, A and B, are shown.

The stiffness and consistent mass matrix formulations were obtained using NASTRAN and were supplied by NASA. A structural damping coefficient of 3% was used for the structure.

The errors assumed for these computations were as follows: 1) $\pm 5\%$ random error on amplitude of accelerations and forces; 2) $\pm 5°$ random phase error on acceleration and force; 3) a random ± 1 in./sec^2 (noise) on transverse acceleration measurements.

The procedure used in these illustrative computations is to shake at each of the constraints.

Response at 88 Hz

The exact responses of the structure at 88 Hz (slightly below the first natural frequency of the free system) due to a force at Station 27 is presented in Fig. 2 for the free condition and for constraints A and B.

None of this information would be available in an actual test. Knowing the "correct" answer is extremely useful when evaluating a suggested test procedure. The simulated tests on each of the constrained systems were run six times at various force levels but using the same nominal errors (as mentioned previously). The results of the deduced free body responses are shown in Fig. 2 as ranges of values.

This figure illustrates two things. First, measurements of the widely differing responses of the system on different supports can be used to determine the response with the constraints removed. Second, it is apparent that different sets of constraints may vary in their sensitivity to measurement error. Notice that the results of using constraints B are more consistent, even though the same nominal errors were used in both sets of simulated tests.

Error Sensitivity

It is possible that the differences in the ranges in results

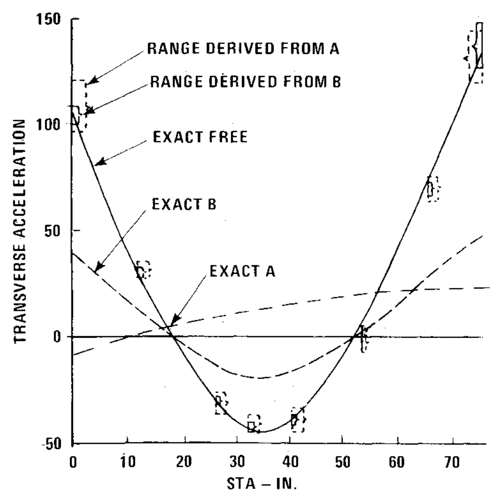


Fig. 2 Transverse bending responses at 88 Hz of two constrained conditions and the deduced free-body responses, including effects of measurement error.

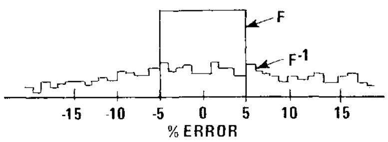


Fig. 3 Error sensitivity of force matrix from constraints A at 88 Hz.

shown on Fig. 2 are simply coincidences because of the relatively small number of simulated tests. In order to more carefully evaluate this condition, the force matrices from these two conditions were subjected to a test of their conditioning. The elements of the force matrices were modified by adding a uniform random error of up to $\pm 5\%$, the matrices were inverted and compared to the exact inverses. This process was repeated several hundred times and the error distributions of the two inverses were recorded and are illustrative in Figs. 3 and 4. This analysis is admittedly rather crude and is only intended to show trends. The amplitude of the elements are not considered here and a large percentage error on a small unimportant element is given the same consideration as if it were on a large and important one. The tendency is for the larger errors to appear with the smaller elements.

It is quite apparent, however, that at this frequency, constraint B is significantly less subject to amplification of error and should yield the most reliable results.

This study illustrates that it may be a very worthwhile process, prior to any test, to perform simulations using approximate characteristics of the structure and to study the sensitivity to errors.

Frequency Response

Figure 5 illustrates the exact response of station 27 to a force at the same station over a frequency range from 80–440 Hz covering the first three nonzero natural frequencies of the free system. Also are shown the deduced ranges of values obtained from the 12 simulated constrained tests, A and B, discussed previously. At frequencies up to 100 Hz, two ranges are shown. In each case but one, the narrower of the two ranges corresponds to the tests using constraints B. This is consistent with the data shown on Fig. 2. At 100 Hz the condition is reversed and the larger errors are associated with B. Above 100 Hz the two ranges are very nearly equal and the extremes from both sets of tests are shown as a single range.

The ranges shown on the figure are extremes and the chances of data falling near these extreme values will be small. A typical test run results in a curve passing through the bracketed regions with most points near the centers of the ranges.

These data are illustrative of the fact that it is possible to obtain reasonably accurate free-body responses from constrained tests. The technique, not unlike any testing methods, will vary in accuracy and reliability depending on the conditions of the test.

Conclusions

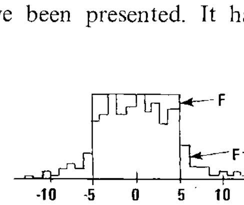
1) A method for obtaining free-body responses using constrained test data only and requiring no assumptions regarding quantitative characteristics of the structure was selected for evaluation as a practical technique.

2) Techniques for implementing this method have been discussed.

3) Illustrative simulated tests have been presented. It has been shown that the sensitivity to error can vary with test conditions. It is suggested that this characteristic is common to many types of tests.

4) The data shown illustrates the potential practicality of the method presented.

Fig. 4 Error sensitivity of force matrix from constraints B at 88 Hz.



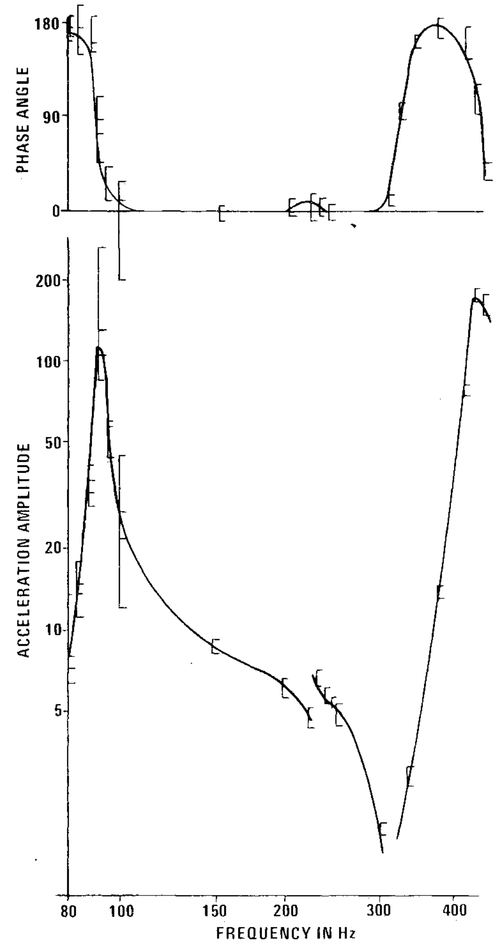


Fig. 5 Frequency response of station 27 showing error bounds.

References

[1] Mixson, J. S. and Catherine, J. J., "Comparison of Experimental Vibration Characteristics Obtained From a 1/5 Scale Model and From a Full-Scale Saturn SA-1," TN D-2215, Nov. 1964, NASA.

[2] Mixson, J. S. and Catherine, J. J., "Experimental Lateral Vibration Characteristics of a 1/5 Scale Model of Saturn SA-1 With an Eight-Cable Suspension System," TN D-2214, Nov. 1964, NASA.

[3] Pinson, L. D. and Leonard, H. W., "Longitudinal Vibration Characteristics of a 1/10-Scale Apollo/Saturn V Replica Model," TN D-5159, April 1969, NASA.

[4] Przemieniecki, J. S., *Theory of Matrix Structural Analysis*, McGraw-Hill, New York, 1968.

[5] Dugundji, J., "On the Calculation of Natural Modes of Free-Free Structures," *Journal of the Aerospace Sciences*, Vol. 28, No. 2, Feb. 1961, pp. 164–166.

[6] Berman, J. H. and Sklerov, J., "Calculation of Natural Modes of Vibration for Free-Free Structures in Three Dimensional Space," *AIAA Journal*, Vol. 3, No. 1, Jan. 1965, pp. 158–160.

[7] Asher, G. W. and Hyland, F. G., "The Use of Admittance Techniques in Dynamic Model Testing," *Proceedings of the AIAA Symposium in Structural Dynamics and Aeroelasticity*, AIAA, New York, 1965.

[8] Berman, A., Giansante, N., and Flannelly, W. G., "Research on Structural Dynamic Testing by Impedance Methods," USAAMRDL TR 72-63C, Nov. 1972, U.S. Army Air Mobility Research and Development Lab., Fort Eustis, Va.

[9] Thornton, E. A., "Vibration Analysis of a 1/15 Scale Dynamic Model of a Space Shuttle Configuration," NASA CR-111984, 1971, Old Dominian Univ., Norfolk, Va.